Section 2.5
Evaluating Limits Algebraically

by Joseph Phillip Brennan
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(1) Determinate and Indeterminate Forms

(2) Limit Calculation Techniques

  (A) Direct Substitution
  (B) Simplification
  (C) Conjugation
  (D) The Squeeze Theorem

(3) Limits of Piecewise-defined and Absolute-Value Functions

The Form of a Limit

The **form** of a limit $\lim_{x \to c} \square$ is the expression resulting from substituting $x = c$ into $\square$.

The form of a limit is **not** the same as its value!
It is a **tool for inspecting** the limit.

$$\lim_{x \to 0} x^{\arctan(x)} : \text{ form } \text{“}0^0\text{”}$$

$$\lim_{x \to \infty} (1+x)^{\frac{1}{x}} : \text{ form } \text{“}\infty^0\text{”}$$

$$\lim_{x \to 0} \cos(x)^{\frac{1}{x}} : \text{ form } \text{“}1^\infty\text{”}$$

$$\lim_{x \to 0^+} \ln(x)\sin(x) : \text{ form } \text{“}-\infty 0\text{”}$$

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The Form of a Limit

Determinate Forms are forms which always represent the same limit. For example, the form "$\frac{1}{\infty}$" always represents a limit which equals 0. Assume $c \neq 0$,

$$\text{“}\frac{c}{0}\text{”} \to \pm\infty \qquad \text{“}\infty \cdot \infty\text{”} \to \infty \qquad \text{“}\infty + \infty\text{”} \to \infty$$

$$\text{“}\frac{c}{\pm\infty}\text{”} \to 0 \qquad \text{“}-\infty \cdot \infty\text{”} \to -\infty \qquad \text{“}-\infty - \infty\text{”} \to -\infty$$

Indeterminate Forms are called indeterminate because they represent limits which may or may not exist and may be equal to any value. The form itself does **not** indicate the value of the limit. There are 7 indeterminate forms:

$$\frac{0}{0} \qquad \pm\frac{\infty}{\infty} \qquad \infty - \infty \qquad \pm 0 \cdot \infty \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0}$$

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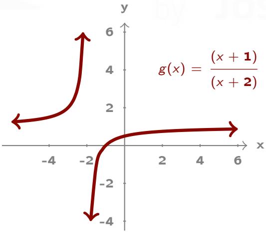
We can use **direct substitution** to evaluate limits of functions that are continuous *(Section 2.4)* or have determinate forms.

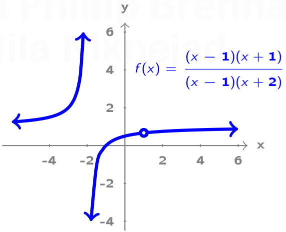
Simplification and Limits

If $f(x) = g(x)$ for values **near** $x = a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

$$\boxed{\frac{(x-1)(x+1)}{(x-1)(x+2)} = \frac{x+1}{x+2} \ \underline{\text{when}} \ x \neq 1}$$





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Conjugation

The expression $a + b$ is **conjugate** to the expression $a - b$.

$$a^2 - b^2 = (a - b)(a + b)$$

Rationalize the denominator:

$$\frac{7}{3 + \sqrt{3}}$$

Rationalize the numerator:

$$\frac{\sqrt{12} - \sqrt{3}}{2}$$

Simplification and Conjugation Examples

(Example I) Evaluate the following limits:

(a) $\displaystyle\lim_{x\to -1} \frac{-2x^2+4x+6}{x^2-x-2}$

(c) $\displaystyle\lim_{t\to 0} \frac{(t+3)^2-9}{t}$

(b) $\displaystyle\lim_{h\to 0} \frac{\sqrt{h^2+4}-2}{h^2}$

(d) $\displaystyle\lim_{x\to -7} \frac{\frac{1}{7}+\frac{1}{x}}{7+x}$

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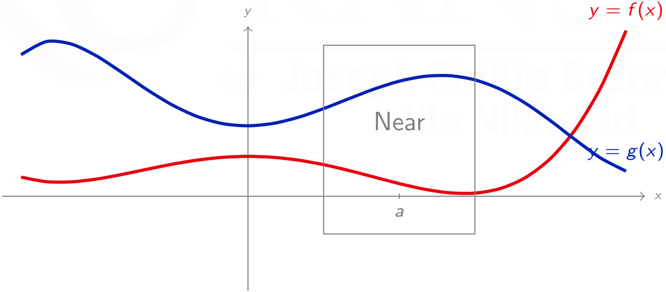
Limits of Comparable Functions

If $f(x) \leq g(x)$ for values of x near a then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

if both limits exist.

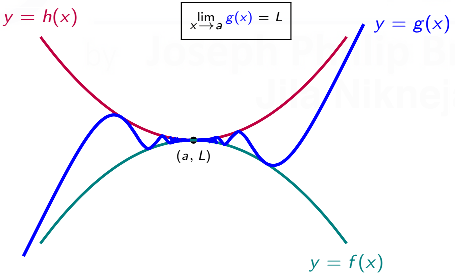


The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ for values of x near a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

then $\lim_{x \to a} g(x) = L$.



“Walking a drunk through a door"

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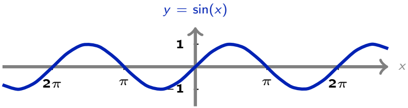
The Squeeze Theorem

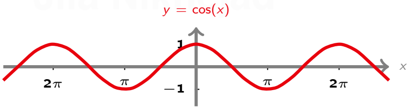
If $f \leq g \leq h$ for values near $x = a$ and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

then $\lim_{x \to a} g(x) = L$.

$$-1 \leq \sin(x) \leq 1 \qquad\qquad -1 \leq \cos(x) \leq 1$$

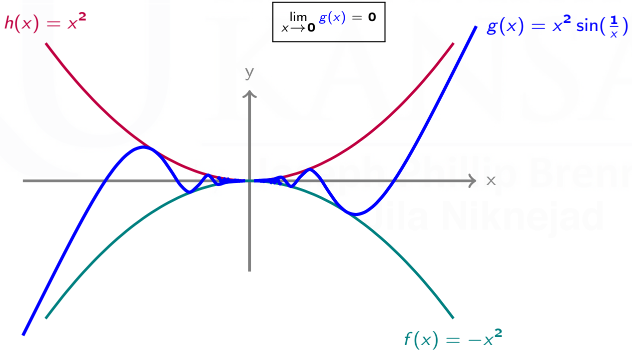




Example II: The Squeeze Theorem

Evaluate

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$



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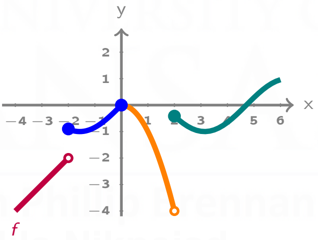
Piecewise-defined and Absolute Value Functions

Evaluating the limit of a piecewise-defined function differs from evaluating the limit of elementary functions only when the limiting value is a **break point**. *($\{-2, 0, 2\}$ below)*

$$f(x) = \begin{cases} x & x < -2 \\ \sin(x) & -2 \le x \le 0 \\ -x^2 & 0 < x < 2 \\ \cos(x) & x \ge 2 \end{cases}$$

$$f(x) = \begin{cases} {\color{purple}x} & x < -2 \\ {\color{blue}\sin(x)} & -2 \le x \le 0 \\ {\color{orange}-x^2} & 0 < x < 2 \\ {\color{teal}\cos(x)} & x \ge 2 \end{cases}$$



Absolute value functions are secretly piecewise functions!

$$|x - a| = \begin{cases} -(x - a) & x < a \\ x - a & x > a \end{cases}$$

When confronted with an absolute valued function, calmly write it as a piecewise function before any other step.